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Merging Higher Derivative Gravity and Quantum Mechanics

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The most general quantum mechanical wave equation for a massive scalar particle in a metric generated by a spherically symmetric mass distribution is considered within the framework of higher derivative gravity (HDG). The exact effective Hamiltonian is constructed and the significance of the various terms is discussed using the linearized version of the above-mentioned theory. Not only does this analysis shed new light on the long standing problem of quantum gravity concerning the exact nature of the coupling between a massive scalar field and the background geometry, it also greatly improves our understanding of the role of HDG's coupling parameters in semiclassical calculations.

KEY WORDS: higher derivative gravity; quantum mechanics; nonminimal coupling.

1. INTRODUCTION

The nonrenormalizability of general relativity (Deser and Niewenhuizen, 1974; Goroff and Sagnotti, 1985; t'Hooft and Veltman, 1974) has inspired the construction of various alternative models for quantum gravity. Among these models there is one that definitely holds a prominent place: higher derivative gravity, or HDG for short. This theory is defined by the Lagrangian density

$$\mathcal{L}_{\mathrm{HDG}} = \sqrt{-g}\left[\frac{2}{\kappa^2}R + \frac{\alpha}{2}R^2 + \frac{\beta}{2}R_{\mu\nu}^2\right], \tag{1}$$

where α and β are dimensionless coupling parameters and $\kappa^2 = 32\pi G$, with G being Newton's constant.

Actually, HDG is famed for being currently the only known gravity theory that is renormalizable along with its coupling constants (Stelle, 1977). Despite being renormalizable, however, HGD possesses a ghost pole in the tree propagator which renders it nonunitary within the standard pertubation scheme. We shall not discuss this problem here, restricting ourselves to draw attention to some pertinent

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references (Antoniadis and Tomboulis, 1986; Fradkin and Tseytlin, 1982; Johston, 1988; Stelle, 1977). Leaving aside the unsolved question of HDG's unitarity—a problem, incidentally, that is a bitter pill for quantum field theorists to swallow—HDG has very attractive properties and can be used, among other things, as an efficient tool in undertaking calculations in which the gravitational field is considered as a classical background field, interacting with quantum matter (Accioly *et al.*, 2000a,b; Accioly and Blas, 2001).

On the other hand, physical phenomena in which gravitational and quantum effects are interwoven, are no more beyond our reach (Bonse and Wroblewski, 1983; Colella and Overhauser, 1975; Nesvizhevsky *et al.*, 2002), which certainly requires that we acquaint ourselves with the issue of the gravitational effects on quantum mechanical systems. In other words, we must learn how to handle relativistic field equations in a curved background spacetime.

Our aim here is to study the effects of HDG, treated as a background classical field, on spin-0 particles with nonvanishing mass. This approach casts light on the important question about the exact nature of the coupling between the massive scalar field and the background geometry. In addition, it makes our understanding of the role of HDG's coupling parameters in semiclassical calculations better.

We discuss the propagation of the spinless massive particle in the geometry generated by a spherically symmetric mass distribution in Section 2, while the exact effective Hamiltonian is constructed in Section 3. In Section 4 we analyze the significance of the various terms of the Hamiltonian using the linearized version of HDG. We conclude in Section 5 with some discussions and comments.

We use natural units throughout. In our convention the signature is $(+\,-\,-\,-)$. The curvature tensor is defined by $R^{\alpha}_{\beta\gamma\delta} = -\partial_\delta T^{\alpha}_{\beta\gamma} + \cdots$, the Ricci tensor by $R_{\mu\nu} = R^{\alpha}_{\mu\nu\alpha}$, and the curvature scalar by $R = g^{\mu\nu} R_{\mu\nu}$, where $g_{\mu\nu}$ is the metric tensor.

2. THE MOST GENERAL COVARIANT KLEIN–GORDON EQUATION IN THE GEOMETRY GENERATED BY A SPHERICALLY SYMMETRIC MASS DISTRIBUTION

Currently, we do not have a standard theory of massive spinless bosons in curved space. That is not the case as far as the Dirac fermions are concerned. Therefore our first task is to generalize the usual Klein–Gordon equation to the case of a spacetime with nonvanishing curvature. Consider in this direction a real, massive scalar field, for which the Lagrangian density is

$$\mathcal{L} = \sqrt{-g}\big[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\big]. \tag{2}$$

Now, the only possible local scalar coupling between ϕ and the gravitational field with the correct dimensions is $\lambda R\phi^2$, where λ is a new coupling constant (Birrell and Davies, 1982). Consequently, we incorporate this term into (2). The resulting Lagrangian as well as the corresponding wave equation are given respectively by

$$\mathcal{L} = \sqrt{-g}\big[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - (m^2 + \lambda R)\phi^2\big], \tag{3}$$

$$(\Box + m^2 + \lambda R)\phi = 0. \tag{4}$$

This equation may be regarded within the context we have just outlined as the most general covariant Klein–Gordon equation in curved space. Note that there are two popular choices for λ: minimal coupling ($\lambda = 0$) and conformal coupling ("conformal coupling" is from now on a loose term for "massive scalar field nonminimally coupled to the curved background with a coupling constant $\lambda = \frac{1}{6}$"). The former leads to the simplest equation of motion, whereas the latter gives rise to a theory which is conformally invariant in the massless limit. For the time being, however, we need not to settle this matter, but rather consider λ on the same footing as m, i.e., as a parameter which specifies the theory.

Since we are interested in the geometry generated by a spherically symmetric mass distribution, we choose to work in Schwarzschild coordinates in isotropic form, in which the invariant interval takes the form

$$ds^2 = [V(r)]^2\, dt^2 - [W(r)]^2[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)], \tag{5}$$

where V^2 and W^2 are solutions of the equations derived from (1). Incidentally, the field equations following from (1), supplemented by a matter Lagrangian density, are

$$\frac{2}{\kappa^2}G_{\mu\nu} + \frac{\alpha}{2}\left[-\frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu} + 2\nabla_\mu\nabla_\nu R - 2g_{\mu\nu}\Box R\right] + \frac{\beta}{2}\left[-\frac{1}{2}g_{\mu\nu}R^2_{\rho\sigma}\right.$$
$$\left. + \nabla_\mu\nabla_\nu R + 2R_{\mu\rho\lambda\nu}R^{\rho\lambda} - \frac{1}{2}g_{\mu\nu}\Box R - \Box R_{\mu\nu}\right] + \frac{1}{2}T_{\mu\nu} = 0. \tag{6}$$

A long but rather straightforward calculation using our coordinates gives the following expression for the Ricci scalar

$$R = 2\frac{(\nabla W)^2}{W^4} - 4\frac{\nabla^2 W}{W^3} - 2\frac{\nabla^2 V}{VW^3} - 2\frac{\nabla V \cdot \nabla W}{VW^3}. \tag{7}$$

Inserting (5) into (4) we find

$$\ddot{\phi} - F^2\nabla^2\phi - F^2\nabla\ \ln(VW)\cdot\nabla\phi + m^2V^2\phi + \lambda RV^2\phi = 0, \tag{8}$$

where $F^2 \equiv \frac{V^2}{W^2}$, R is given by Eq. (7) and the dots denote differentiation with respect to time. The above is the most general covariant Klein–Gordon equation in a general static classical background within the framework of HDG.

3. KLEIN–GORDON EQUATION IN SCHRÖDINGER FORM

To understand the physics of Eq. (8) is advantageous to rewrite it in Schrödinger formalism. In other words, we have to transform the Klein–Gordon equation into a system of two coupled differential equations that are of the first-order in time. This is achieved by the ansatz

$$\phi = a + b, \qquad \frac{i\dot{\phi}}{m} = a - b.$$

The two coupled differential equations

$$i\dot{a} = \left(\frac{m}{2} - \Lambda\right) a - \left(\Lambda + \frac{m}{2}\right) b, \tag{9}$$

$$i\dot{b} = \left(\frac{m}{2} + \Lambda\right) a + \left(\Lambda - \frac{m}{2}\right) b, \tag{10}$$

where

$$\Lambda \equiv \frac{1}{2m}[F^2\nabla^2 + F^2\nabla\ \ln(VW)\cdot\nabla - m^2V^2 - \lambda R V^2],$$

are equivalent to Eq. (8). Indeed, the addition of Eqs. (9) and (10) leads to the trivial equation $\dot{\phi} = \dot{\phi}$, while its subtraction reproduces (8).

Introducing the column vector

$$\Phi = \begin{pmatrix} a \\ b \end{pmatrix} \tag{11}$$

and making use of the 2×2 matrices

$$\tau = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad \tau^{\mathrm{T}} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},$$

that fulfill the algebraic relations

$$\tau^2 = \mathcal{O}, \qquad \{\tau, \tau^{\mathrm{T}}\} = 4I,$$

where

$$\mathcal{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we obtain a Schrödinger-type equation, namely,

$$i\dot{\Phi} = \mathcal{H}\Phi,$$

with the Hamiltonian given by

$$\mathcal{H} = \frac{m}{2}\tau^{\mathrm{T}} - \tau\Lambda.$$

Of course, $\mathcal{H}$ is not a Hermitian operator; its square,

$$\mathcal{H}^2 = -\frac{m}{2}\{\tau, \tau^{\mathrm{T}}\}\Lambda = -2m\Lambda I,$$

nevertheless, is diagonal. Therefore, formally,

$$\mathcal{H} = \sqrt{\mathcal{H}^2} = (-2m\Lambda)^{1/2} I^{1/2}. \tag{12}$$

Now, the operator Λ is Hermitian when the requisite spatial integrations are carried out using the correct measure (Fulling, 1991)

$$\langle\Lambda\rangle = \int \rho\, d^3\mathbf{x}\psi^*\Lambda\psi,$$

where $\rho \equiv g^{00}\sqrt{-g} = \frac{W^3}{V}$.

However, it is more convenient to absorb ρ into the wave functions ψ and ψ^* in order that Λ appears as a Hermitian operator when integrated with respect to the Euclidian coordinates $\int d^3\mathbf{x}$. This can be achieved by introducing a new wave function ψ',

$$\psi' = \rho^{1/2}\psi = V^{-1/2}W^{3/2} \equiv f\psi,$$

and a corresponding Hamiltonian $\mathcal{H}' = f\mathcal{H}f^{-1}$, in terms of which Eq. (12) becomes

$$\mathcal{H}' = \sqrt{\mathcal{H}'^2} = (-2m\Lambda')^{1/2} I^{1/2},$$

where

$$\begin{aligned}\Lambda' &= f\Lambda f^{-1}\\ &= -\frac{1}{2m}(m^2V^2 + \lambda RV^2) + \frac{F^2}{2m}\left\{\left[-2\frac{\nabla f}{f} + \nabla\ln(VW)\right]\cdot\nabla\right.\\ &\quad\left. -\frac{\nabla^2 f}{f} + 2\left(\frac{\nabla f}{f}\right)^2 + \nabla^2 - \nabla\ln(VW)\cdot\frac{\nabla f}{f}\right\}.\end{aligned} \tag{13}$$

If we take into account that

$$\frac{\nabla f}{f} = \frac{\nabla F}{F} + \frac{1}{2}\nabla\ln(VW),$$

we promptly obtain the useful results

$$\left(\frac{\nabla f}{f}\right)^2 = \left(\frac{\nabla F}{F}\right)^2 - \frac{1}{F}\nabla F\cdot\nabla\ln(VW) + \frac{1}{4}[\nabla\ln(VW)]^2,$$

$$\frac{\nabla^2 f}{f} = 2\left(\frac{\nabla F}{F}\right)^2 + \frac{1}{4}[\nabla\ln(VW)]^2 + \frac{1}{2}\nabla^2\ln(VW) - \frac{\nabla F}{F}\cdot\nabla\ln(VW) - \frac{\nabla^2 F}{F}.$$

Inserting the above into Eq. (13), we get

$$\Lambda' = -\frac{m}{2}V^2 - \frac{\lambda R V^2}{2m} + \frac{F^2}{2m}\left\{2\frac{\nabla F}{F}\cdot\nabla + \nabla^2 + \frac{\nabla^2 F}{F} - \frac{1}{4}[\nabla \ln(VW)]^2 - \frac{1}{2}\nabla^2 \ln(VW)\right\}. \tag{14}$$

Using the identity

$$F^2\nabla^2 \equiv -F\hat{\mathbf{p}}^2 F - F\nabla^2 F - 2F\nabla F\cdot\nabla,$$

where $\hat{\mathbf{p}} = \frac{\nabla}{i}$ is the momentum operator, we may rewrite Eq. (14) as

$$\Lambda' = -\frac{mV^2}{2} - \frac{1}{2m}F\hat{\mathbf{p}}^2 F + \frac{1}{8m}\nabla F\cdot\nabla F - \frac{1}{2m}\mathcal{D}_\lambda(V, W),$$

with

$$\mathcal{D}_\lambda(V, W) \equiv \lambda\left[\left(\frac{1}{2\lambda} - 2\right)\frac{V}{W^2}\nabla^2 V - 2\frac{V}{W^3}\nabla V\cdot\nabla W + \left(\frac{1}{2\lambda} - 4\right)\frac{V^2}{W^3}\nabla^2 W + 2\frac{V^2}{W^4}(\nabla W)^2\right]. \tag{15}$$

Consequently,

$$\mathcal{H}' = \left[m^2V^2 + F\hat{\mathbf{p}}^2 F - \frac{1}{4}\nabla F\cdot\nabla F + \mathcal{D}_\lambda(V, W)\right]^{1/2} I^{1/2}. \tag{16}$$

The matrix $I^{1/2}$ appearing in the Hamiltonian above must be dealt with some care. Indeed, the square root of the 2×2 identity matrix is not unique. On the other hand, the eigenvalues of any matrix whose square is I are ± 1. For our purposes, however, it is convenient to choose $I^{1/2}$ as a 2×2 matrix with distinct eigenvalues. In this case it may be diagonalized by a similarity transformation. In other words, there exists a 2×2 invertible matrix P such that

$$\eta = P^{-1}I^{1/2}P,$$

where

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

As a consequence, P is also a diagonalizing matrix for $\mathcal{H}'$, i.e., $\overline{\mathcal{H}} = P^{-1}\mathcal{H}'P$ is diagonal. Therefore, $\overline{\mathcal{H}}$ assumes the form

$$\overline{\mathcal{H}} = \begin{pmatrix} \mathcal{H}_{\mathrm{FW}} & 0 \\ 0 & -\mathcal{H}_{\mathrm{FW}} \end{pmatrix},$$

where

$$\mathcal{H}_{\text{FW}} = \left[m^2 V^2 + F\hat{\mathbf{p}}^2 F - \frac{1}{4}\nabla F \cdot \nabla F + \mathcal{D}_\lambda(V, W) \right]^{1/2}. \tag{17}$$

Accordingly, we have succeeded in transforming the Hamiltonian for a Klein–Gordon particle into a form where positive and negative energy states are decoupled. The transformation $\mathcal{H} \to \overline{\mathcal{H}}$ is thus the Foldy–Wouthuysen transformation for the most general Klein–Gordon equation in curved space within the context of HDG. Of course, the effective Hamiltonian, $\mathcal{H}_{\text{FW}}$, defined by Eq. (17) is nothing but the Foldy–Wouthuysen Hamiltonian. It is quite remarkable that the Foldy–Wouthuysen transformation we have constructed is *exact*. It is worth mentioning that it exists for *any* λ as well.

4. ANALYZING THE NONRELATIVISTIC LIMIT OF THE THEORY

As we have seen in the last Section, $\mathcal{H}_{\text{FW}}$ is *exact*. Nonetheless, we are generally interested in situations where it is sufficient to use nonrelativistic wave functions, treating the gravitational interaction as an ordinary pertubation in nonrelativistic quantum mechanics. The effective quasirelativistic Hamiltonian is then trivially obtained by assuming that the m term in Eq. (17) is the dominating term and correspondingly expanding the square root of (17) as

$$\mathcal{H}_{\text{FW}} \approx mV + \frac{1}{4m}(W^{-1}\hat{\mathbf{p}}^2 F + F\hat{\mathbf{p}}^2 W^{-1}) - \frac{1}{8mV}\nabla F \cdot \nabla F + \frac{\mathcal{D}_\lambda(V, W)}{2mV}. \tag{18}$$

Before going on, it is necessary to find the classical background field. Far away from the central gravitating body of a mass M and in the Teyssandier gauge (Teyssandier, 1989), the metric is given by (see, for instance, Accioly *et al.*, 2000a)

$$g_{00} = 1 - 2\Phi_{\text{N}} \left(1 + \frac{1}{3}\, e^{-m_0 r} - \frac{4}{3}\, e^{-m_1 r} \right), \tag{19}$$

$$g_{ii} = -1 - 2\Phi_{\text{N}} \left(1 - \frac{1}{3}\, e^{-m_0 r} - \frac{2}{3}\, e^{-m_1 r} \right), \tag{20}$$

where $\Phi_{\text{N}} \equiv \frac{MG}{r}$ is the Newtonian potential, $i = 1, 2, 3$, $m_0^2 \equiv \frac{2}{\kappa^2(3\alpha+\beta)}$ and $m_1^2 \equiv -\frac{4}{\kappa^2\beta}$. Note that we are assuming that $m_1^2 > 0\,(\beta < 0)$ and $m_0^2 > 0$ $(3\alpha + \beta > 0)$, which corresponds to the absence of tachyons (both positive and negative energy) in the dynamical field. From Eqs. (19) and (20) we get immediately

$$V \approx 1 - A\Phi_{\text{N}}, \qquad W \approx 1 + B\Phi_{\text{N}}, \qquad F \approx 1 - C\Phi_{\text{N}}, \tag{21}$$

where

$$A \equiv 1 + \frac{1}{3}\, e^{-m_0 r} - \frac{4}{3}\, e^{-m_1 r},$$

$$B \equiv 1 - \frac{1}{3}\, e^{-m_0 r} - \frac{2}{3}\, e^{-m_1 r},$$

$$C \equiv A + B = 2\,(1 - e^{-m_1 r}).$$

Asymptotically, A, $B \to 1$ and $C \to 2$. In this case, (19) and (20) reduces to the linearized Schwarzschild solution in isotropic form concerning general relativity. We are now ready to analyze the quasirelativistic Hamiltonian Eq. (18). For simplicity's sake we consider first its asymptotical limit. In this case,

$$V \to 1 - \Phi_{\mathrm{N}}, \qquad W \to 1 + \Phi_{\mathrm{N}}, \qquad F \to 1 - 2\Phi_{\mathrm{N}}. \tag{22}$$

From Eqs. (18) and (22), we come to the conclusion that

$$\mathcal{H}_{\mathrm{FW}} \to m + m\mathbf{g}\cdot\mathbf{r} + \frac{3}{2m}\hat{\mathbf{p}}\cdot(\mathbf{g}\cdot\mathbf{r})\hat{\mathbf{p}} + \frac{\hat{\mathbf{p}}^2}{2m} + \frac{1}{m}\left(\frac{3}{4} - \lambda\right)\nabla^2\Phi_{\mathrm{N}}, \tag{23}$$

where $\boldsymbol{g} \equiv -\frac{MG}{r^3}\mathbf{r}$. Note that we have neglected the higher order relativistic and gravitational terms.

For the Sun, for instance, $\Phi_{\mathrm{N}} \sim 10^{-6}$, which implies that the term $\frac{3}{2m}\hat{\mathbf{p}}\cdot(\mathbf{g}\cdot\mathbf{r})\hat{\mathbf{p}}$ is less than the kinetic term by a factor of $\sim 10^{-6}$ and much weaker by several orders than the leading and next to leading terms. As a consequence, we shall not take the contribution of this term into account. Thus,

$$\mathcal{H}_{\mathrm{FW}} \to m + \frac{\hat{\mathbf{p}}^2}{2m} + m\mathbf{g}\cdot\mathbf{r} + \frac{1}{m}\left(\frac{3}{4} - \lambda\right)\nabla^2\Phi_{\mathrm{N}}, \tag{24}$$

The first two terms in Eq. (24) give the usual expression for the relativistic energy of the massive spin-0 particle correct to order v^2, where v is the particle's velocity, while the third one gives the Newtonian contribution. Interesting enough the quantum mechanical acceleration, namely,

$$\mathbf{a} = -[H, [H, \mathbf{r}]],$$

yields the usual Newtonian result

$$\mathbf{a} = \nabla\Phi_{\mathrm{N}},$$

if we employ the simple form

$$H = m + \frac{\hat{\mathbf{p}}^2}{2m} + m\mathbf{g}\cdot\mathbf{r}.$$

The last term, which following Obukhov (Obukhov, 2001) we shall call the *gravitational Darwin term* (GDT), may be attributed to the zitterbewegung. Because the boson's position fluctuates an amount δr such that $\langle \delta r^2 \rangle \approx \frac{1}{m^2}$, it sees a somewhat smeared out Newtonian potential; the correction in the effective gravitostatic energy is

$$\langle m\delta\Phi_{\rm N}\rangle = \langle m\Phi_{\rm N}(\mathbf{r}+\delta\mathbf{r})\rangle - \langle m\Phi_{\rm N}(\mathbf{r})\rangle \approx \frac{1}{6m}\nabla^2\Phi_{\rm N}. \tag{25}$$

The GDT we have found, namely, $\frac{1}{m}(\frac{3}{4}-\lambda)\nabla^2\Phi_{\rm N}$, is in qualitative accord with Eq. (25) as far as the sign, form, and magnitude are concerned, if $0 \leq \lambda < \frac{3}{4}$, where we have assumed as usual that the coupling constant should be nonnegative. Therefore, the exact effective Hamiltonian exists for any λ; however, for $0 \leq \lambda < \frac{3}{4}$, the GDT is in good agreement with (25).

But, far away from the source and in the approximation considered the contribution of the GDT is zero. Thus,

$$\mathcal{H}_{\rm FW} \to m + \frac{\hat{\mathbf{p}}^2}{2m} + m\mathbf{g}\cdot\mathbf{r}. \tag{26}$$

where $\mathbf{g} \equiv -\frac{GM}{r^3}\mathbf{r}$.

We discuss now whether or not the nonminimal coupling violates the equivalence principle. This can be easily done by comparing the true gravitational coupling with the purely inertial case. For the flat Minkowski space in accelerated frame,

$$V = 1 + \mathbf{a}\cdot\mathbf{r}, \qquad W = 1, \qquad F = W,$$

and we get

$$\mathcal{H}_{\rm FW} \approx m + \frac{\hat{\mathbf{p}}^2}{2m} + m\mathbf{a}\cdot\mathbf{r}, \tag{27}$$

where we have neglected the higher order relativistic and inertial terms. It follows from Eqs. (26) and (27) that the nonminimal coupling, unlike it is commonly believed, does not violate the equivalence principle.

To conclude we analyze Eq. (18) using Eq. (21). The effective Hamiltonian is now given by

$$\mathcal{H}_{\rm FW} \approx m - mA\Phi_{\rm N} - \frac{1}{2m}\hat{\mathbf{p}}\cdot(B+C)\Phi_{\rm N}\hat{\mathbf{p}} + \frac{\hat{\mathbf{p}}^2}{2m} + \nabla^2\left[\frac{A+C}{4m} - \frac{\lambda(2B-A)}{m}\right]\Phi_{\rm N}. \tag{28}$$

On the other hand, it was shown recently that semiclassical HDG leads to dispersive photon propagation (Accioly and Blas, 2001). In other words, gravitational rainbows and semiclassical HDG can coexist without conflict. On the basis of the fact that the rainbow effect is currently undetectable, it is possible to show that $|\beta| \leq 10^{60}$ (Accioly and Blas, 2001). Consequently, we assume that $|\alpha| \approx |\beta| \sim 10^{60}$. Therefore, $\overline{m} \equiv m_1 \approx m_0 \sim 10^3$ cm^{-1}, which tells us that for the Sun, for instance, $\bar{m}R_{\odot} \sim 10^{13}$. So, we come to the conclusion that the corrections due to HDG quadratic terms, i.e., R^2 and $R^2_{\mu\nu}$, are essentially smaller than what the nowadays experimental technique can detect; as a result, Eq. (28) reduces, in practice, to Eq. (26).

5. DISCUSSIONS AND COMMENTS

There are some arguments in the literature that seem to favor the conformal coupling ($\lambda = \frac{1}{6}$) to the detriment of other couplings. A number of claims of these works were recently rehearsed (Accioly and Blas, 2002) and carefully tackled (Accioly and Blas, 2003). In short, we may say that the aforementioned assertions are in general inconclusive. Our calculations, however, show that $0 \leq \lambda < \frac{3}{4}$, which includes, of course, the conformal coupling in the list of the possible couplings. Note that for this particular coupling, Eq. (15) leads to

$$\mathcal{D}_{1/6}(V, W) = \frac{1}{6} F \nabla^2 F,$$

since

$$F\nabla^2 F \equiv \frac{V}{W^2}\nabla^2 V - 2\frac{V}{W^3}\nabla V \cdot \nabla W - \frac{V^2}{W^3}\nabla^2 W + 2\frac{V^2}{W^4}(\nabla W)^2,$$

implying that the GDT is given by

$$\frac{F\nabla^2 F}{12mV} - \frac{1}{4m}(W^{-1}\nabla^2 F + F\nabla^2 W^{-1}).$$

Undoubtedly, in this case the GDT has the simplest structure, while for other values of λ its structure is more complicated. If we appeal to simplicity as a guide, as is usually done in Physics, we certainly should single out the conformal coupling. Nonetheless, this criterion is neither orthodox nor robust. In truth, it seems more of an experimental problem to identify which would be the correct λ coupling(s) for the various scalar particles. Furthermore, the presence of the GDT endowed with the simplest structure is certainly a too mild argument in support of the conformal coupling. Anyway, our computations, at least in principle, bound the coupling constant without the detriment of the traditional choices: $\lambda = 0$ and $\lambda = \frac{1}{6}$. Its astonishing, nevertheless, that the exact effective Hamiltonian exists for any λ; moreover, there is no conflict between the nonminimal coupling and the equivalence principle.

As far as the influence of the quadratic terms of HDG on the effective Hamiltonian is concerned, all we can say is that this effect cannot be observed because it is beyond the present experimental reach. Certainly, Einstein's gravity is currently the appropriate theory to deal with these effects.

Last but not least, we call attention to the fact that that even today there are some physicists who believe that the Foldy–Wouthuysen transformation only exists for Dirac particles. Of course, this idea must be dismissed as a complete nonsense. For a discussion concerning the Foldy–Wouthuysen transformation related to the Klein–Gordon equation in flat space see, for instance, the works of Case (1954) and Feshbach and Villars (1958), as well as the well-known textbook by Bjorken and Drell (1964). Incidentally, if the gravitational field is "switched off" ($V^2 = W^2 = 1$), Eq. (17) reduces to $\sqrt{m^2 + \hat{\mathbf{p}}^2}$, which is precisely the *exact* Foldy–Wouthuysen Hamiltonian for the Klein–Gordon equation in flat space found by Bjorken and Drell (1964) using a rather different approach.

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